Some methods for flows past blunt slender bodies

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It is suggested that the use of prolate spheroidal co-ordinates in certain problems involving slender bodies may lead to results which not only are more likely to be uniformly valid for blunt bodies, but in many cases require less complicated analysis than results obtained by standard methods which use cylindrical co-ordinates. The method is developed for a simple problem in potential theory and is then applied also to a problem in Stokes flow, yielding a procedure for obtaining the Stokes drag on a slender body of arbitrary shape. For comparison purposes, consideration is also given to the use of both cylindrical and di-polar co-ordinates, and as a by-product of the comparison of results on cylindrical and spheroidal systems some new simple formulae involving Legendre polynomials are obtained heuristically, and then rigorously proved.

1. Introduction

The purpose of the present paper is to introduce some new techniques which may prove useful in treating a variety of problems where a basic flow is disturbed by a slender (i.e. needle-like) body. The emphasis on clear description of techniques means that the problems treated in this paper are highly idealized, but it is expected that extensions to more practical flows would involve only a little extra effort in conception or manipulation. Thus we consider only bodies of revolution in a uniform stream which is parallel to their axis and is of infinite extent, and we assume incompressibility. We then investigate only the two formal limiting versions of the Navier–Stokes equations, i.e. that limit of zero viscosity which gives irrotational potential flow, and the limit of infinite viscosity giving Stokes flow.

Possible generalizations of these idealized cases are unlimited, and some are of great practical importance; we need merely mention amongst such generalizations linear slender body problems in acoustic, optical or water-wave diffraction, cavitating flow, Oseen flow, subsonic or supersonic aerodynamics, steady ship motion, and many other fields. Besides extension to other physical conditions, the methods may also be applied with other geometrical configurations. Thus for instance the extension to the case of non-axisymmetry is possible in every case with increase only in manipulative, not conceptual, difficulty. Also, most of the work may be carried over by analogy to treat the case of thin (i.e. wafer-like, or nearly planar, as distinct from needle-like, or nearly linear) bodies.

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The main proposal of this paper is that, wherever practicable, slender body theory be attacked via prolate spheroidal rather than cylindrical co-ordinates. The question of choice of co-ordinate system is always a bothersome one, and of course one always tries to formulate any problem in applied mathematics in a manner independent of the system chosen. But the fact remains that slender body theory is a co-ordinate perturbation process where, in the 'classical' approach (e.g. Thwaites 1960, p. 387) the assumption is made that the body lies in a region for which the cylindrical co-ordinate r is small in some sense. Now there is no *a priori* reason for preferring cylindrical co-ordinates, and there soon appear some good *a posteriori* reasons why this is a bad choice.

The main motivation for considering alternative formulations of slender body theory is the fact that the approach via cylindrical co-ordinates leads immediately to difficulties at the ends of a finite blunt body, such as velocities wrongly predicted to be infinite rather than zero. There are of course already methods (in thin aerofoil theory, see again Thwaites 1960) enabling corrections to be made for this unpleasant behaviour, but these still retain the cylindrical co-ordinate basis, and it was thought that a new approach might be more fruitful. The defects in the cylindrical co-ordinate approach are more serious in the ship problem (Vossers 1962; Tuck 1963) and in Stokes flow than in aerodynamic problems, and the end-effects introduced into these problems by use of these co-ordinates seem to be irremovable.

The two co-ordinate systems tried in this paper are prolate spheroidal and di-polar, although only the first of these is seriously recommended as a practical proposition. In fact, it became apparent that the analysis required for the use of spheroidal co-ordinates is often less involved than that required for cylindrical co-ordinates, so that they may be recommended even when end-effects are not present. In a sense, of course, the end-effects from cylindrical co-ordinates are always present, for even if the body is cusped so that no infinities are predicted, we may expect that the numerical accuracy of the theory is bad in any region of high curvature. One would perhaps expect from a naïve point of view that the above mentioned co-ordinates would be more appropriate than cylindrical co-ordinates for finite bodies, since the co-ordinate surfaces include examples (prolate spheroids and spindles, respectively) of such bodies. There are better reasons than this, however, and these are discussed in the text.

On the other hand, it is of course necessary to find a set of solutions to the governing partial differential equation in prolate spheroidal co-ordinates, and this may not always be possible or easy in cases more complicated than those treated in this paper (although for most of the extensions mentioned earlier it is certainly possible). One also lacks techniques like Fourier transforming in the streamwise direction, although if an integral transform involving Legendre functions is needed the one discovered by Clemmow (1961) may prove useful.

Nevertheless, there still remains a large class of practical linear (or linearizable) problems for which the use of spheroidal co-ordinates is possible and not difficult; the ease with which the results on Stokes flow are obtained in § 7 may be taken to illustrate this.

As a by-product of the treatment of slender body potential theory by spheroidal co-ordinates in § 3, some new results have been discovered concerning Legendre polynomials. In effect, the result obtained is an expansion for the reciprocal $1/|x_1 - x_2|$ of the distance between two points x_1, x_2 on the real line between ± 1 in terms of Legendre polynomials of arguments x_1 and x_2 (the modulus sign is significant; analogous expansions for $1/(x_1 - x_2)$ are of course well known; Erdelyi 1953, §§ 3.10.10 and 3.14.7). This expansion is typical of a class of expansions which may be obtained heuristically by comparing slender body theories in various co-ordinate systems with that using cylindrical co-ordinates; an expansion in conical functions is suggested from the work on di-polar co-ordinates but is not explicitly found in the present work. A rigorous proof by induction of the result on Legendre polynomials is given in an Appendix.

2. Slender body potential theory via cylindrical co-ordinates

Slender body theory is often taken to be a branch of supersonic aerodynamics, since it is in this field that most of its applications have appeared. However, the formal differences between supersonic and subsonic slender body theory are slight (for example, Laplace transforms are used instead of Fourier transforms) and the subsonic theory is a simple generalization of incompressible theory, so there is some justification in neglecting compressibility in an exposition of techniques. Thwaites (1960) has given a treatment of incompressible theory which shows the strong analogy with the supersonic theory presented by Ward (1949).

One way of looking at 'classical' slender body theory (but not the only, or necessarily the most profitable way) is to imagine the body replaced by a line distribution of sources. This is logical if the problem is viewed as an asymptotic one, since when the body shrinks down to its limiting axis the fluid occupies the whole space except for the axis itself, and the velocity potential must be an analytic function everywhere except *on* this axis. To fix ideas, let us set up a cylindrical co-ordinate system (x, r) with the uniform stream U in the positive x direction. Then we shall assume that the total velocity potential may be written in the form

$$\phi = Ux - \frac{1}{2}\int_{-\infty}^{\infty} d\xi \frac{a(x-\xi)}{(\xi^2 + r^2)^{\frac{1}{2}}} \tag{2.1}$$

for some, so far arbitrary, function $a(x)$ describing the density of the source distribution. For small but non-zero slenderness, equation (2.1) requires that the potential can be continued analytically inside the body as far as the axis, an assumption that is frequently unjustified, but let us for the moment assume that there is a class of bodies for which it is justified.

Now it can be shown, either by Fourier transforming the disturbance potential (Thwaites, p. 387) or directly from the form (2.1) (Goldstein 1960, p. 184; see also Appendix I), that for small r

$$\phi = Ux + a(x)\log r + b(x) + E, \tag{2.2}$$

where the error E tends to zero as $r \downarrow 0$, and where

$$b(x) = -\frac{1}{2}\int_{-\infty}^{\infty} da(\xi)\,\mathrm{sgn}\,(x-\xi)\log 2|x-\xi|. \tag{2.3}$$

In fact, if the approximation (2.2) is to hold uniformly in x, certain regularity conditions must be satisfied by $a(x)$, and in Appendix I it is shown rigorously that if $a(x)$ is bounded and continuous, has a bounded and piecewise continuous derivative, and is (absolutely) integrable over the whole real x-axis, then the error term E is *at most* of order $r^{\frac{1}{2}}$ for small r, uniformly with respect to x. By considering further terms in the asymptotic expansion (the so-called 'inner expansion') begun by (2.2), it is easy to see that the *least* possible order of E is $r^2 \log r$, but this would only be achieved uniformly if derivatives of $a(x)$ existed everywhere to all orders.

The function $a(x)$ is still arbitrary, and is to be found from the boundary condition on the body surface. In the axisymmetric case, if the body is described by the equation

$$r = r_0(x)$$

then the boundary condition of zero normal velocity requires

$$\frac{\partial \phi}{\partial r} = r_0'(x)\frac{\partial \phi}{\partial x} \quad \text{on} \quad r = r_0(x), \tag{2.4}$$

whence (2.2) gives, to highest order in r_0,

$$a(x) = U r_0(x)\, r_0'(x) \tag{2.5}$$

$$= (U/2\pi)\, S'(x), \tag{2.6}$$

where $S(x)$ is the area of cross-section. It can easily be shown that the result (2.6) holds also for non-axisymmetric bodies.

In practice, the above regularity conditions on $a(x)$ therefore mean that the body surface itself, as defined by $r_0(x)$ in the present case, must be smooth. But even if the internal surface is smooth to a high order, there will still usually be trouble at the ends (unless the body is infinitely long, in which case the integrability condition requires its slope to tend to zero suitably rapidly). In fact, from (2.5), we see that the above conditions are violated at an end $x = 0$ if $r_0(x)$ does not vanish at least as fast as x, so that for instance a blunt body with finite curvature at its ends violates the conditions.

Of course the above conditions are sufficient for the potential only, and do not guarantee small errors on differentiation of (2.2) to give velocities. This is well illustrated by calculating the velocity magnitude on the body in the form

$$q^2 = \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right]_{r=r_0(x)}$$

$$= U^2 + U^2 r_0'^2 + 2Ua' \log r_0 + 2Ub' + O(r_0^4 \log^2 r_0) \tag{2.7}$$

(the order of the error term given is that of the next term in the asymptotic series and is not therefore a uniform bound in general). Clearly for a blunt body such that $r_0 \sim x^{\frac{1}{2}}$, we have $q^2 \to \infty$ like $1/x$ near $x = 0$, instead of vanishing as it should to give a stagnation point. But also in the case when $r_0 \sim x$, it follows that $a'(x) \sim 1$ so that q^2 behaves like $\log x$ near $x = 0$; that is, the velocity is wrongly predicted to tend to infinity logarithmically near the nose of a cone-shaped body.

These end-effect troubles have long been known in thin aerofoil theory, and there are methods (often empirical) for avoiding them. However, Lighthill (1951) has proposed a systematic method of treating blunt aerofoils using a co-ordinate stretching method, and his method may be carried over almost unchanged to slender-body theory. Nevertheless, we now suggest two different approaches which achieve similar ends in a more straightforward manner.

3. Slender body potential theory via spheroidal co-ordinates

It is clear that the end-effect troubles described in the previous section arose simply because we chose to work with an unsuitable co-ordinate system. Thus we apply the body surface boundary condition in a region of small r such that $\partial/\partial r$ is large compared with $\partial/\partial x$ (when applied to the disturbance potential $\phi - Ux$); that is, we are approximating normal derivatives by radial derivatives. But this cannot be a sensible thing to do near the end of a blunt body, where the normal is in the x, not the r, direction. Lighthill's method distorts the cylindrical co-ordinates in the neighbourhood of the ends in such a way that the normal direction is correctly approximated by the distorted r direction, and small modifications to formulae like (2.7) can be derived from his method to eliminate the difficulties of end-effects.

However, a more natural approach is to discard the cylindrical co-ordinates altogether in favour of systems which fit the physical situation better. We have chosen to discuss the use of either prolate spheroidal or di-polar co-ordinates; other choices are possible, but these give the simplest results.

Prolate spheroidal co-ordinates (ξ, η) may be defined by their relationship with cylindrical co-ordinates as follows:

$$\left.\begin{aligned} x &= l\cos\xi\cosh\eta, \\ r &= l\sin\xi\sinh\eta, \end{aligned}\right\} \tag{3.1}$$

where l is a representative length (see e.g. Lamb 1932, p. 139; Hobson 1931, p. 142). Sometimes it is convenient to write $\mu = \cos\xi$, $\zeta = \cosh\eta$, and use (μ, ζ) as co-ordinates; even more convenient in the present context is the 'mixed' combination (μ, η) which we shall adopt henceforth. Surfaces $\mu =$ constant are hyperboloids of revolution with foci at $x = \pm l$, $r = 0$. Similarly, surfaces of constant η are prolate spheroids with the same foci; $\eta = 0$ is that part of the x-axis between $x = \pm l$, while $\eta = +\infty$ is the surface at infinity.

It is clear that η is somewhat analogous to r, and in particular that if η is small then r is necessarily small (although the converse is not true). Thus for η small, we have

$$\left.\begin{aligned} x &= l\mu + O(\eta^2), \\ r &= l(1-\mu^2)^{\frac{1}{2}}\eta + O(\eta^3), \end{aligned}\right\} \tag{3.2}$$

i.e. for small η, μ becomes the non-dimensional x co-ordinate x/l, while

$$\eta \to r(l^2 - x^2)^{-\frac{1}{2}}.$$

Now suppose we have a body whose equation in spheroidal co-ordinates is

$$\eta = \eta_0(\mu)$$

with η_0 small. Then, approximately, the equation in terms of cylindrical co-ordinates must be

$$r = r_0(x) = \begin{cases} \eta_0(x/l)\,(l^2 - x^2)^{\frac{1}{2}} & (|x| < l), \\ 0 & (|x| > l). \end{cases} \tag{3.3}$$

Thus if η_0 tends to finite values near $\mu = \pm 1$, then the body has finite curvature at the ends. If η_0 behaves like $(1-\mu^2)^{\frac{1}{2}}$ the body has cone-shaped ends, while if η_0 tends to zero more rapidly than this the body is cusped.

Typical solutions of Laplace's equation in these co-ordinates which vanish at infinity are multiples of

$$P_n(\mu)\,Q_n(\zeta)$$

(Lamb 1932, p. 140), where P_n and Q_n are the usual solutions of Legendre's equation. Let us assume that the potential for flow past the slender body $\eta = \eta_0(\mu)$ can be written as a linear combination of these functions in the form

$$\phi = Ux - Ul \sum_{n=0}^{\infty} \alpha_n Q_n(\cosh\eta)\,P_n(\mu), \tag{3.4}$$

for some sequence of real numbers α_n. In order to apply the boundary condition on the body surface we require a knowledge of the behaviour of this potential for small values of η. Now for small η,

$$-Q_n(\cosh\eta) = \log\tfrac{1}{2}\eta + \sigma_n + O(\eta^2\log\eta), \tag{3.5}$$

where

$$\sigma_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + 1/n \quad (\sigma_0 = 0) \tag{3.6}$$

(Erdelyi 1953, equation 3.9.7). Hence the inner expansion begins

$$\phi = Ux + Ul\alpha(\mu)\log\tfrac{1}{2}\eta + Ul\beta(\mu) + O(\alpha\eta^2\log\eta), \tag{3.7}$$

where

$$\alpha(\mu) = \sum_{n=0}^{\infty} \alpha_n P_n(\mu), \tag{3.8}$$

and

$$\beta(\mu) = \sum_{n=0}^{\infty} \sigma_n \alpha_n P_n(\mu). \tag{3.9}$$

The functional relationship between $\alpha(\mu)$ and $\beta(\mu)$ is implicitly exhibited by the pair of equations (3.8) and (3.9), and the best method of calculating $\beta(\mu)$ for given $\alpha(\mu)$ would usually be to calculate the Fourier–Legendre coefficients α_n first. However, if desired, the relationship may be shown as an explicit integral, for, on using the orthogonality relations for the P_n, we have

$$\beta(\mu) = \frac{1}{2}\int_{-1}^{1} d\mu'[\alpha(\mu) - \alpha(\mu')]\,K(\mu,\mu'), \tag{3.10}$$

where the kernel $K(\mu,\mu')$ takes the form

$$K(\mu,\mu') = -\sum_{n=0}^{\infty} (2n+1)\,\sigma_n P_n(\mu)\,P_n(\mu'). \tag{3.11}$$

This series will be summed explicitly later.

Equation (3.7) bears a striking resemblance to equation (2.2), and this is no coincidence. Notice that here also the coefficient of the logarithm of one co-ordinate is an arbitrary function of the other co-ordinate, while the 'constant term' is a definite functional of this coefficient.

Now the inner boundary condition on the slender body $\eta = \eta_0(\mu)$ is

$$\frac{\partial \phi}{\partial \eta} = (1-\mu^2)\,\eta_0'(\mu)\frac{\partial \phi}{\partial \mu} \quad \text{on} \quad \eta = \eta_0(\mu).$$

Hence, on substituting equation (3.7) and retaining terms of highest order in η_0, we have

$$\begin{aligned}\alpha(\mu) &= (1-\mu^2)\,\eta_0(\mu)\,\eta_0'(\mu) - \mu\eta_0^2(\mu)\\ &= d/d\mu\,[\tfrac{1}{2}(1-\mu^2)\,\eta_0^2(\mu)].\end{aligned} \tag{3.12}$$

But this is equivalent to equation (2.6) since by (3.3) we have approximately

$$Ul\alpha(\mu) = U\frac{d}{dx}[\tfrac{1}{2}r_0^2(x)] = \frac{U}{2\pi}S'(x) = a(x).$$

Thus once again the coefficient of the logarithm is $U/2\pi$ times the streamwise derivative of the cross-sectional area. This is not surprising, since approximately

$$\log \tfrac{1}{2}\eta = \log r - \log 2(l^2-x^2)^{\frac{1}{2}},$$

so that from (3.7) we can re-derive the inner expansion in cylindrical co-ordinates. We should also expect that the constant terms are the same, which implies that

$$b(x) = Ul\beta(x/l) - a(x)\log 2(l^2-x^2)^{\frac{1}{2}}.$$

Since both $b(x)$ and $\beta(\mu)$ are defined by integral functionals of $a(x) = Ul\alpha(\mu)$, this in turn implies a relationship between the kernels in equations (2.3) and (3.10).

Now in the present case when $a(x)$ is identically zero for $|x| > l$, integration of (2.3) by parts gives $|x| < l$:

$$b(x) = \frac{1}{2}\int_{-l}^{l} d\xi\,\frac{a(x)-a(\xi)}{|x-\xi|} - a(x)\log 2(l^2-x^2)^{\frac{1}{2}}.$$

Hence
$$Ul\beta\left(\frac{x}{l}\right) = \frac{1}{2}\int_{-l}^{l} d\xi\,\frac{a(x)-a(\xi)}{|x-\xi|},$$

or
$$\beta(\mu) = \frac{1}{2}\int_{-1}^{1} d\mu'\,\frac{\alpha(\mu)-\alpha(\mu')}{|\mu-\mu'|}.$$

Thus we have obtained the result

$$K(\mu, u') = \frac{1}{|\mu-\mu'|} = -\sum_{n=0}^{\infty}(2n+1)\,\sigma_n\,P_n(\mu)\,P_n(\mu'). \tag{3.13}$$

This expansion appears to be new in the theory of Legendre polynomials. By orthogonality, equation (3.13) implies and is implied by the integral formula

$$\int_{-1}^{1} d\mu'\,\frac{P_n(\mu)-P_n(\mu')}{|\mu-\mu'|} = 2\sigma_n\,P_n(\mu), \tag{3.14}$$

which also appears to be a new result. Since the above argument leading to these results is a circuitous one incidental to the main objectives of this section, a direct proof of (3.14) by induction is given in Appendix II.

Returning to the slender body problem, let us for purposes of comparison evaluate again the velocity magnitude on the surface of the body. The exact formula may easily be seen to be

$$q^2 = \frac{1-\mu^2}{l^2(\zeta^2-\mu^2)}[1+(1-\mu^2)\,\eta_0'^2]\left(\frac{\partial\phi}{\partial\mu}\right)^2,$$

which has the inner expansion

$$\frac{q^2}{U^2} = \frac{1-\mu^2}{1-\mu^2+\eta_0^2(\mu)}[1+(1-\mu^2)\,\eta_0'^2+2\alpha'\log\tfrac{1}{2}\eta_0+2\beta'+O(\eta_0^4\log^2\eta_0)]. \quad (3.15)$$

Notice that we have not been quite consistent in retaining the term $\eta_0^2(\mu)$ in the denominator rather than expanding in a binomial series. This artifice, however, enables the formula to become uniformly valid even for a blunt body, for if η_0 takes finite values near $\mu = \pm 1$ then q^2 vanishes there to give the required stagnation point. This is the type of modified formula which can be derived from Lighthill's theory and other theories for blunt aerofoils.

For a cone-shaped end, where $\eta_0 \sim (1-\mu^2)^{\frac{1}{2}}$, α' takes finite values at the ends, so that just as for the cylindrical co-ordinates expression (2.7), the formula (3.15) is not uniformly valid since it predicts logarithmically infinite velocity at the ends via the term $2\alpha'\log\frac{1}{2}\eta_0$. There is, therefore, a case for considering a further choice of co-ordinate system suited to this type of end, such as is given in the following section. For a cusped body finite values of q^2 are predicted everywhere, but it is not certain that the theory is valid for such bodies; this question could be settled by an investigation of error bounds analogous to that in Appendix I, but it is thought in any case that the theory of this section would find its chief application to blunt bodies.

As a special case we may consider flow past an exact prolate spheroid $\eta_0 =$ constant. Then from (3.12) we have

$$\alpha(\mu) = -\mu\eta_0^2,$$

from which equations (3.8) and (3.9) give

$$\beta(\mu) = -\mu\eta_0^2.$$

Hence

$$\frac{q^2}{U^2} = \frac{1-\mu^2}{1-\mu^2+\frac{1}{2}\eta_0^2}[1-2\eta_0^2\log\tfrac{1}{2}\eta_0-2\eta_0^2].$$

This expression may be shown to agree with the inner expansion of the exact expression calculated from Lamb's (1932, p. 141) formula for the exact velocity potential. However, it may be wise at this point to emphasize the trivial point that because spheroidal co-ordinates are used in this section we do not by any means imply that the theory holds only for spheroids, any more than the theory of the previous section holds only for cylinders! All theories in this paper are designed for slender bodies of arbitrary shape.

4. Slender body potential theory via di-polar co-ordinates

Di-polar co-ordinates (ξ, η) may be defined by the equations

$$x = \frac{l \sinh \xi}{\cosh \xi + \cos \eta}, \quad r = \frac{l \sin \eta}{\cosh \xi + \cos \eta} \tag{4.1}$$

(Hobson, 1931, p. 449; Payne 1952; we use a slightly different notation which is more convenient for slender body purposes than that of Hobson). Once again η is analogous to r, and small η implies small r.

The details of the analysis for a slender body theory in these co-ordinates will not be reproduced here as some formidable manipulation providing little insight is involved. For a slender body described by

$$\eta = \eta_0(\xi),$$

the inner expansion is found to be

$$\phi = Ux + Ul\alpha(\xi) \log \tfrac{1}{2}\eta + Ul\beta(\xi) + O(\alpha\eta^2 \log \eta), \tag{4.2}$$

where

$$\alpha(\xi) = \tfrac{1}{2}(1+\cosh \xi) \frac{d}{d\xi} \frac{\eta_0^2(\xi)}{(1+\cosh \xi)^2} \tag{4.3}$$

$$\sim \frac{1}{2\pi l} S'(x),$$

and

$$\beta(\xi) = \tfrac{1}{2}(1+\cosh \xi)^{\frac{1}{2}} \int_0^\infty \frac{d\sigma}{\sinh \frac{1}{2}\sigma} [e^{-\frac{1}{2}\sigma}\alpha_1(\xi) - \tfrac{1}{2}\alpha_1(\xi+\sigma) - \tfrac{1}{2}\alpha_1(\xi-\sigma)], \tag{4.4}$$

with

$$\alpha_1(\xi) = \alpha(\xi)(1+\cosh \xi)^{-\frac{1}{2}}.$$

The resulting velocity magnitude on the body is

$$\frac{q^2}{U^2} = 1 + \eta_0'^2 - \eta_0^2(1+\cosh \xi)^{-1} + 2(1+\cosh \xi)[\alpha' \log \tfrac{1}{2}\eta_0 + \beta'] + O(\eta_0^4 \log \eta_0). \tag{4.5}$$

For the special case of an exact spindle, $\eta_0 =$ constant, it can be verified that this result agrees with a calculation of q^2/U^2 based upon an expression for the Stokes stream function given by Payne (1952).

In general the use of di-polar co-ordinates requires much more complicated analysis than that needed for either cylindrical or spheroidal co-ordinates (for instance, the formula (4.4) involves a very unpleasant integration), and the method can hardly be recommended except where it is absolutely necessary to avoid the logarithmic end effects suffered by the other two methods for cone-shaped ends. In fact, if a minimum of calculation is the only criterion for deciding between the three methods and end-effects are not important, then spheroidal co-ordinates appear to be the best and the method of this section is the least satisfactory of the three. There are other factors to be taken into account, of course, the most important being the need to find a complete set of basic solutions in more general problems, and in this regard di-polar co-ordinates are again unsuitable. For instance, the scalar wave equation is separable in spheroidal co-ordinates but not in di-polar co-ordinates.

5. The Stokes drag of a spheroid

Payne & Pell (1960) have given the following exact formula for the drag on a prolate spheroid $\eta = \eta_0 =$ constant, when set with its axis parallel to the stream U in steady Stokes flow

$$D = \frac{8\pi\mu^* Ul}{(1+\cosh^2\eta_0)\log\coth\frac{1}{2}\eta_0 - \cosh\eta_0} \tag{5.1}$$

(with μ^* as the coefficient of viscosity). The properties of this formula are worth investigating in their own right, and also as a guide to what should happen in the general case.

Now if η_0 is small we can approximate equation (5.1) in two stages, giving

$$D = -\frac{8\pi\mu^* Ul}{1+2\log\frac{1}{2}\eta_0}[1+O(\eta_0^2)] \tag{5.2}$$

$$= -\frac{4\pi\mu^* Ul}{\log\frac{1}{2}\eta_0}\left[1+O\left(\frac{1}{\log\eta_0}\right)\right]. \tag{5.3}$$

At the first stage, represented by equation (5.2), only rational approximations have been made, and the error is a factor of order η_0^2, while to obtain (5.3) we have expanded (5.2) further in powers of $1/\log\eta_0$. One would expect that a formula such as (5.2) might be of some practical utility, but that (5.3) would be of no value quantitatively (one has, for instance, doubts as to whether to write the denominator as $\log\frac{1}{2}\eta_0$ or $\log\eta_0$). All (5.3) tells us is the semi-qualitative result that as the body shrinks down to its axis the drag vanishes like $-4\pi\mu^* Ul$ times the reciprocal logarithm of slenderness, with 'slenderness' only vaguely defined (and in particular, arbitrary to the extent of a constant factor).

These features of the drag of a slender spheroid are reproduced in the results for an arbitrary slender body. We should hope as our main objective to derive a formula like (5.2) which would have quantitative value (this difficult aim is only partly achieved in the sequel), and also to obtain a semi-qualitative result like (5.3). The last objective is a comparatively easy task, and we shall be able in fact to prove that the result quoted above for spheroids holds in general.

Other properties of the formula (5.1), though incidental to the main purpose of this paper, are of interest. Thus it is possible to show that at constant volume

$$(\tfrac{4}{3}\pi l^3\cosh\eta_0\sinh^2\eta_0)$$

or at constant surface area

$$(2\pi l^3\sinh\eta_0(\sinh\eta_0+\cosh^2\eta_0\sin^{-1}\operatorname{sech}\eta_0))$$

there exists a unique non-trivial prolate spheroid which minimizes the drag. In the case of constant volume this occurs at $\eta_0 \sim 0{\cdot}575$; since this is of moderate slenderness, it is interesting to try minimizing (5.2) at constant volume, which gives $\eta_0 \sim 0{\cdot}27$. At constant surface area the true minimal spheroid is somewhat more slender ($\eta_0 \sim 0{\cdot}26$) so that the agreement with the slender body approximation ($\eta_0 \sim 0{\cdot}16$) is slightly better. These results, while not showing close quantitative agreement, at least suggest that one possible use of a slender body theory for an arbitrary body shape is in seeking an approximation to the shape which gives least drag. Another method of tackling this minimization problem would be to find the drag on a body which is nearly spheroidal and this has recently been done by the author in unpublished work.

6. Slender body Stokes flow via cylindrical co-ordinates

By analogy with § 2 we might attempt a slender body theory for Stokes flow by representing the body by line distributions of sources and Stokeslets.† Thus we could write the stream function as

$$\psi = \tfrac{1}{2}Ur^2 - \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\xi\,\xi a_1'(x-\xi)}{(\xi^2+r^2)^{\frac{1}{2}}} - \tfrac{1}{2}r^2\int_{-\infty}^{\infty} \frac{d\xi\, a_2(x-\xi)}{(\xi^2+r^2)^{\frac{1}{2}}} \tag{6.1}$$

for some choice of the arbitrary functions $a_1'(x)$ (describing the density of sources) and $a_2(x)$ (describing the density of Stokeslets). Under similar regularity conditions on a_1 and a_2 the inner expansion may be derived from the results of Appendix I, giving

$$\psi = \tfrac{1}{2}Ur^2 + a_1(x) + (a_2(x) - \tfrac{1}{2}a_1''(x))r^2\log r$$
$$+ (b_2(x) - \tfrac{1}{2}b_1''(x) + \tfrac{1}{4}a_1''(x))r^2 + O(a_1 r^4 \log r,\ a_2 r^4 \log r), \tag{6.2}$$

where b_1, b_2 are calculated from a_1, a_2 by equation (2.3). The boundary conditions $\psi = 0$, $\partial\psi/\partial r = 0$ on $r = r_0(x)$ give respectively

$$a_1 = O(a_2 r_0^2 \log r_0), \tag{6.3}$$

$$2a_2(x)\log r_0(x) + a_2(x) + 2b_2(x) + U = 0. \tag{6.4}$$

Terms in a_1, b_1 have been dropped from (6.4) as an immediate consequence of (6.3).

Equations (6.3) and (6.4) imply that $a_2 = O(\log r_0)^{-1}$, $a = O(r_0^2)$ so that as the body shrinks to its limiting axis the source distribution has negligible effect compared with the Stokeslet distribution, which vanishes only slowly and gives rise to the inverse logarithmic behaviour of the drag. Equation (6.4) is a singular integral equation (when $b_2(x)$ is written out as an integral functional of $a_2(x)$) of a particularly awkward variety which nevertheless might repay further study (perhaps numerical). Since only rational approximations (cf. equation (5.2)) have been made in its derivation, one might try expanding in a series of powers of $1/\log r_0$, but this is a futile operation since not only are the successive approximations of no practical use, but they will in general not satisfy the regularity conditions of Appendix I, which are required for the validity of (6.4). Another approach would be to try solving the integral equation by expanding the unknown function $a_2(x)$ in terms of a suitable complete set of orthogonal functions; if Legendre polynomials were used for this we should expect results similar to those to be obtained by direct methods in the next section.

† Distributions of sources and Stokeslets have been used to represent slender bodies in a rather different context by Hancock (1953).

7. Slender body Stokes flow via spheroidal co-ordinates

Since cylindrical co-ordinates have proved even less suitable for Stokes flow than they were for potential flow, let us try spheroidal co-ordinates. Following Payne & Pell (1960) we represent the flow past the slender body $\eta = \eta_0(\mu)$ by the expansion

$$\psi = \tfrac{1}{2}Ur^2\left[1-\sum_0^\infty (\alpha_n P_n(\mu)\, Q_n(\zeta)+A_n\, P'_n(\mu)\, Q'_n(\zeta))\right] \tag{7.1}$$

for some choice of the sequences α_n, A_n. The drag is produced entirely by the contribution of the term in α_0 which behaves at infinity like a Stokeslet and gives a drag of $4\pi\mu^* Ul\alpha_0$. Now, using again the expansion (3.5) for $Q_n(\cosh\eta)$, we have as the inner expansion

$$\psi = \tfrac{1}{2}Ul^2(1-\mu^2)\,\eta^2\left[1+\alpha(\mu)\log\tfrac{1}{2}\eta+\beta(\mu)+O(\alpha\eta^2\log\eta)+\frac{A(\mu)}{\eta^2}+O(A\log\eta)\right], \tag{7.2}$$

where
$$\alpha(\mu) = \sum_{n=0}^\infty \alpha_n P_n(\mu),$$

$$\beta(\mu) = \sum_{n=0}^\infty \sigma_n\alpha_n P_n(\mu),$$

and
$$A(\mu) = \sum_{n=0}^\infty A_n P_n(\mu).$$

Now the boundary conditions

$$\psi = \frac{\partial\psi}{\partial\eta} = 0 \quad \text{on} \quad \eta = \eta_0(\mu)$$

give
$$A(\mu) = \tfrac{1}{2}\alpha(\mu)\,\eta_0^2(\mu), \tag{7.3}$$

and
$$1+\alpha(\mu)\log\tfrac{1}{2}\eta_0(\mu)+\tfrac{1}{2}\alpha(\mu)+\beta(\mu) = 0. \tag{7.4}$$

In terms of the original coefficients α_n, equation (7.4) states that

$$1+\sum_{n=0}^\infty \alpha_n P_n(\mu)\log\tfrac{1}{2}\eta_0(\mu)+\frac{1}{2}\sum_{n=0}^\infty \alpha_n P_n(\mu)+\sum_{n=0}^\infty \sigma_n\alpha_n P_n(\mu) = 0, \tag{7.5}$$

or, on multiplying by $P_m(\mu)$ and integrating,

$$\sum_{n=0}^\infty L_{mn}\alpha_n = -2\delta_{m0}, \tag{7.6}$$

where the matrix elements L_{mn} take the form

$$L_{mn} = \frac{1+2\sigma_m}{2m+1}\delta_{mn}+\int_{-1}^1 d\mu\, P_n(\mu) P_m(\mu)\log\tfrac{1}{2}\eta_0(\mu), \tag{7.7}$$

δ_{mn} being the Kronecker delta function.

Thus, instead of the integral equation (6.4) of the previous section, the use of spheroidal co-ordinates leads directly to an infinite set of linear equations in an infinite number of unknowns α_n. The problem is now effectively solved, since it is only necessary to evaluate the matrix elements L_{mn} for any given $\eta_0(\mu)$ and to invert the matrix equation (7.6) (presumably on a computer).† We are in general interested only in the first coefficient α_0 since this gives the drag, but it is necessary to solve the complete system (7.6) to find α_0. The matrix L_{mn} is somewhat simplified if $\log \frac{1}{2}\eta_0(\mu)$ is a polynomial (of degree k, say) since then $L_{mn} = 0$ for $|m-n| > k$. In particular, for the true spheroid when $\eta_0 = \text{constant}$, L_{mn} becomes wholly diagonal, and (7.6) has the solution

$$\alpha_0 = -\frac{2}{1+2\log \frac{1}{2}\eta_0},$$

$$\alpha_n = 0 \quad (n = 1, 2, 3, \ldots).$$

so that the drag is

$$D = -\frac{8\pi\mu^* Ul}{1+2\log \frac{1}{2}\eta_0}$$

in agreement with (5.2).

Since (7.6) is the result of making only rational approximations, we may investigate the effect of further expanding in powers of order $1/\log \eta_0$. Then, from (7.5) we have

$$\sum_{n=0}^{\infty} \alpha_n P_n(\mu) = -\frac{1}{\log \frac{1}{2}\eta_0(\mu)} + O\left(\frac{1}{\log \eta_0}\right)^2, \tag{7.8}$$

or in particular

$$\alpha_0 = -\frac{1}{2}\int_{-1}^{1} \frac{d\mu}{\log \frac{1}{2}\eta_0(\mu)} + O\left(\frac{1}{\log \eta_0}\right)^2.$$

But if $\frac{1}{2}\eta_0(\mu) = \epsilon\eta_1(\mu)$, where ϵ is a small parameter describing slenderness and $\eta_1(\mu)$ is of order unity with respect to ϵ, then

$$(\log \tfrac{1}{2}\eta_0(\mu))^{-1} = (\log \epsilon)^{-1} + O(\log \epsilon)^{-2},$$

so that

$$\alpha_0 = -(\log \epsilon)^{-1} + O(\log \epsilon)^{-2},$$

giving a drag

$$D = -\frac{4\pi\mu^* Ul}{\log \epsilon} + O\left(\frac{1}{\log \epsilon}\right)^2.$$

Thus as $\epsilon \to 0$ the drag tends to zero like $-4\pi\mu^* Ul$ times the reciprocal of the logarithm of slenderness. This result has only qualitative meaning, since ϵ is arbitrary to the extent of a constant factor.

Appendix I

In this Appendix we investigate the behaviour for small r of the expression

$$\Phi = -\frac{1}{2}\int_{-\infty}^{\infty} d\xi\, a(x-\xi)\,(\xi^2+r^2)^{-\frac{1}{2}} \tag{AI. 1}$$

† This is of course an over-simplification of the numerical difficulties that might arise in solving (7.6). Since the matrix is infinite, we must either replace the infinite upper limit by as large an integer as computer capacity allows, or else use an iterative procedure. The question of convergence must then be considered, but while this is an interesting problem in itself since conditions on $\eta_0(\mu)$ for numerical convergence would be a guide to the class of bodies for which the theory is valid, it is not intended to pursue the point further in this paper.

for functions $a(x)$ which are bounded and continuous, possess bounded and piecewise continuous derivatives everywhere, and are absolutely integrable. Let us put

$$\Phi = \int_{-\delta(r)}^{\delta(r)} + \int_{\delta(r)}^{\infty} + \int_{-\infty}^{-\delta(r)}$$

for some $\delta(r) > 0$. Now consider

$$E_1 = -\frac{1}{2}\int_{-\delta(r)}^{\delta(r)} d\xi\,[a(x-\xi)-a(x)]\,(\xi^2+r^2)^{-\frac{1}{2}}. \tag{AI. 2}$$

Then
$$|E_1| \leqslant \frac{\delta}{r}\{\text{max. of } |a(x-\xi)-a(x)| \quad \text{in} \quad |\xi| \leqslant \delta\}$$
$$\leqslant \frac{\delta^2}{r}\{\text{max. of } |a'(x-\xi)| \quad \text{in} \quad |\xi| \leqslant \delta\}.$$

Thus
$$E_1 = O(\delta^2/r) \tag{AI. 3}$$

uniformly in x, since $a'(x)$ is bounded in every δ-neighbourhood of every x by the above regularity conditions.

Secondly, consider

$$E_2 = -\frac{1}{2}\int_{\delta(r)}^{\infty} d\xi\, a(x-\xi)\,\{(\xi^2+r^2)^{-\frac{1}{2}}-\xi^{-1}\}. \tag{AI. 4}$$

Then
$$|E_2| \leqslant \frac{1}{2}\int_{\delta}^{\infty} d\xi\,|a(x-\xi)|\left|\frac{\xi-(\xi^2+r^2)^{\frac{1}{2}}}{\xi(\xi^2+r^2)^{\frac{1}{2}}}\right|$$
$$\leqslant \frac{1}{2}\,\frac{(\delta^2+r^2)^{\frac{1}{2}}-\delta}{\delta(\delta^2+r^2)^{\frac{1}{2}}}\int_0^{\infty} d\xi\,|a(x-\xi)|.$$

Similarly if
$$E_3 = -\frac{1}{2}\int_{-\infty}^{-\delta(r)} d\xi\, a(x-\xi)\,\{(\xi^2+r^2)^{-\frac{1}{2}}+\xi^{-1}\}, \tag{AI. 5}$$

then
$$|E_3| \leqslant \frac{1}{2}\,\frac{(\delta^2+r^2)^{\frac{1}{2}}-\delta}{\delta(\delta^2+r^2)^{\frac{1}{2}}}\int_{-\infty}^{0} d\xi\,|a(x-\xi)|.$$

Thus
$$|E_2+E_3| \leqslant \frac{1}{2}\,\frac{(\delta^2+r^2)^{\frac{1}{2}}-\delta}{\delta(\delta^2+r^2)^{\frac{1}{2}}}\int_{-\infty}^{\infty} d\xi\,|a(x-\xi)|$$
$$\leqslant \frac{1}{4}\,\frac{r^2}{\delta^3}\int_{-\infty}^{\infty} d\xi\,|a(\xi)|,$$

i.e.
$$E_2+E_3 = O(r^2/\delta^3) \tag{AI. 6}$$

uniformly in x, since $a(x)$ is absolutely integrable.

Hence by (AI. 2), (AI. 4) and (AI. 5), we have

$$\Phi = -\tfrac{1}{2}a(x)\int_{-\delta}^{\delta}\frac{d\xi}{(\xi^2+r^2)^{\frac{1}{2}}}+E_1-\frac{1}{2}\int_{\delta}^{\infty}\frac{d\xi\,a(x-\xi)}{\xi}+E_2+\frac{1}{2}\int_{-\infty}^{-\delta}\frac{d\xi\,a(x-\xi)}{\xi}+E_3, \tag{AI. 7}$$

where the error bounds (AI. 3) and (AI. 6) give

$$E_1+E_2+E_3 = E_4 = O(\delta^2/r)+O(r^2/\delta^3). \tag{AI. 8}$$

But $\delta(r)$ is as yet arbitrary, and we may choose it in such a way that this total error is small, with $\delta = o(r^{\frac{1}{2}})$ but $r = o(\delta^{\frac{3}{2}})$. The 'best' order of magnitude of δ is given by

$$\delta(r) = O(r^{\frac{3}{5}}),$$

whereupon

$$E_4 = O(r^{\frac{1}{5}}),$$

since any other order for δ gives a larger order for E_4.

Now, on integrating the first term of (AI. 7) explicitly and the second and third terms by parts, we have

$$\Phi = -\tfrac{1}{2}a(x)\log\frac{\delta+(\delta^2+r^2)^{\frac{1}{2}}}{-\delta+(\delta^2+r^2)^{\frac{1}{2}}}+\tfrac{1}{2}\{a(x-\delta)+a(x+\delta)]\log\delta$$
$$-\frac{1}{2}\int_\delta^\infty d\xi\{a'(x-\xi)-a'(x+\xi)\}\log\xi+E_4. \qquad \text{(AI. 9)}$$

But if

$$E_5 = -\tfrac{1}{2}a(x)\log\frac{\delta+(\delta^2+r^2)^{\frac{1}{2}}}{-\delta+(\delta^2+r^2)^{\frac{1}{2}}}+\tfrac{1}{2}a(x)\log\frac{4\delta^2}{r^2}$$
$$= -\tfrac{1}{2}a(x)\log[\tfrac{1}{2}+\tfrac{1}{2}(1+r^2/\delta^2)^{\frac{1}{2}}+\tfrac{1}{2}r^2/\delta^2], \qquad \text{(AI. 10)}$$

then

$$|E_5| \leqslant \tfrac{1}{2}|a(x)|\,.\tfrac{3}{4}r^2/\delta^2 = O(r^{\frac{4}{5}}) = o(r^{\frac{1}{5}}),$$

while if

$$E_6 = \tfrac{1}{2}\{a(x-\delta)+a(x+\delta)\}\log\delta-a(x)\log\delta$$
$$= \tfrac{1}{2}[\{a(x+\delta)-a(x)\}-\{a(x)-a(x-\delta)\}\log\delta], \qquad \text{(AI. 11)}$$

then

$$|E_6| \leqslant \tfrac{1}{2}|\log\delta|\{|a(x+\delta)-a(x)|+|a(x)-a(x-\delta)|\}$$
$$\leqslant \delta|\log\delta|\{\text{max. of }|a'(x)|\},$$

so that

$$E_6 = O(r^{\frac{3}{5}}\log r) = o(r^{\frac{1}{5}}),$$

and, finally, if

$$E_7 = \frac{1}{2}\int_0^\delta d\xi\{a'(x-\xi)-a'(x+\xi)\}\log\xi, \qquad \text{(AI. 12)}$$

then

$$|E_7| \leqslant \frac{1}{2}\int_0^\delta d\xi\{|a'(x-\xi)|+|a'(x+\xi)|\}|\log\xi|$$
$$\leqslant \{\text{max. of }|a'(x)|\}\int_0^\delta d\xi\,|\log\xi|$$
$$= O(\delta\log\delta) = o(r^{\frac{1}{5}}).$$

Thus

$$\Phi = -\tfrac{1}{2}a(x)\log 4x^2/r^2+E_5-a(x)\log\delta+E_6$$
$$-\int_0^\infty d\xi\{a'(x-\xi)-a'(x+\xi)\log\xi\}+E_7+E_4 \qquad \text{(AI. 13)}$$
$$= a(x)\log r+b(x)+E, \qquad \text{(AI. 14)}$$

where $b(x)$ is as given by (2.3) (after change of variable), and where we have now proved that, for the class of functions $a(x)$ described at the beginning of this Appendix, the error E is at most of order $r^{\frac{1}{5}}$ uniformly with respect to x. Clearly by further restricting the class of permitted $a(x)$ we could find smaller error bounds; however, since the next term is the asymptotic expansion of Φ can easily be seen to be of order $r^2\log r$ no non-trivial restrictions on $a(x)$, however severe, can produce an error term smaller than $O(r^2\log r)$.

Appendix II

We wish to prove by induction the result (3.14); that is, that if

$$f_n(x) = \frac{1}{2}\int_{-1}^{1} dt \frac{P_n(x)-P_n(t)}{|x-t|}, \tag{AII. 1}$$

then

$$f_n(x) = \sigma_n P_n(x). \tag{AII. 2}$$

Now by use of the recurrence relation (Erdelyi 1953, 10.10.9) for Legendre polynomials, we have from (AII. 1):

$$(n+1)f_{n+1}(x) = \tfrac{1}{2}(2n+1)\int_{-1}^{1} dt \frac{xP_n(x)-tP_n(t)}{|x-t|} - \tfrac{1}{2}n\int_{-1}^{1} dt\frac{P_{n-1}(x)-P_{n-1}(t)}{|x-t|}$$

$$= -nf_{n-1}(x)+(2n+1)xf_n(x)+\tfrac{1}{2}(2n+1)\int_{-1}^{1} dt\,\mathrm{sgn}\,(x-t)P_n(t);$$

i.e. f_n satisfies the inhomogeneous Legendre recurrence relation

$$\begin{aligned}(n+1)&f_{n+1}(x)-(2n+1)xf_n(x)+nf_{n-1}(x)\\ &= \frac{2n+1}{2}\int_{-1}^{x} P_n(t)\,dt - \frac{2n+1}{2}\int_{x}^{1} P_n(t)\,dt\\ &= \frac{1}{2}\int_{-1}^{x} dt\,[P'_{n+1}(t)-P'_{n-1}(t)] - \frac{1}{2}\int_{x}^{1} dt\,[P'_{n+1}(t)-P'_{n-1}(t)]\\ &= P_{n+1}(x)-P_{n-1}(x).\end{aligned} \tag{AII. 3}$$

Now if we assume the truth of (AII. 2) for $n \leqslant m$, then

$$\begin{aligned}(m+1)&f_{m+1}(x)\\ &= (2m+1)\,x\sigma_m P_m(x) - m\sigma_{m-1}P_{m-1}(x)+P_{m+1}(x)-P_{m-1}(x)\\ &= (m+1)\sigma_{m+1}P_{m+1}(x)-\sigma_m[(m+1)P_{m+1}(x)-(2m+1)xP_m(x)+mP_{m-1}(x)]\\ &\quad -(m+1)\,[\sigma_{m+1}-\sigma_m]\,P_{m+1}(x)+P_{m+1}(x)+m[\sigma_m-\sigma_{m-1}]\,P_{m-1}(x)-P_{m-1}(x)\\ &= (m+1)\,\sigma_{m+1}P_{m+1}(x),\end{aligned}$$

since the first square bracket vanishes by the Legendre recurrence relation and the second takes the value $(m+1)^{-1}$ by the definition (3.6) of the σ function.

Thus (AII. 2) is true for $n = m+1$, if true for $n \leqslant m$. Since it is true by inspection for $n = 0$ and $n = 1$, this completes the proof by induction.

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CORRIGENDUM

'Departures from the linear equations for vibrational relaxation in shock waves in oxygen and carbon dioxide', by H. K. ZIENKIEWICZ and N. H. JOHANNESEN, *J. Fluid Mech.* **17**, 1963, pp. 499–506.

In the note added in proof on page 505, the word 'smallest' in the last line should read 'largest'.